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High-temperature series analysis of the p-state Potts glass model on d-dimensional hypercubic lattices

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Abstract. We analyze recently extended high-temperature series expansions for the "Edwards-Anderson" spin-glass susceptibility of the p-state Potts glass model on d-dimensional hypercubic lattices for the case of a symmetric bimodal distribution of ferro- and antiferromagnetic nearest-neighbor couplings $J_{ij} = \pm J$. In these star-graph expansions up to order 22 in the inverse temperature $K \equiv J\beta \equiv J/k_BT$, the number of Potts states p and the dimension d are kept as free parameters which can take any value. By applying several series analysis techniques to the new series expansions, this enabled us to determine the critical coupling K_c and the critical exponent γ of the spin-glass susceptibility in a large region of the two-dimensional (p, d)-parameter space. We discuss the thus obtained information with emphasis on the lower and upper critical dimensions of the model and present a careful comparison with previous estimates for special values of p and d.

PACS. 75.10.Nr Spin-glass and other random models – 75.10.Lk Spin-glasses and other random magnets – 64.60.Fr Equilibrium properties near critical points, critical exponents

1 Introduction

Spin-glass models are used to describe quenched, disordered materials with randomly distributed, competing interactions [1–4]. While the latter property is the characteristic feature of all spin-glasses, for specific applications also the spin degrees of freedom are an important ingredient of the model. The most extensively studied prototype model is the Ising spin-glass where each spin can only take the two different values $S_i = \pm 1$. A generalization to p discrete states per spin, $S_i \in \{1, \ldots, p\}$ is the Potts spin-glass [5–8], which can be considered as the generic model of anisotropic orientational glasses [9]. Materials of this type arise from random dilution of molecular crystals such as N_2 diluted with Ar [10]. Here the model parameter p is associated with the p orientations of the uniaxial molecule in the crystal. Typical cases are $p = 3$, when the molecules can align only along the x, y, and z axes of a cubic crystal, and $p = 6$, when the face diagonals are the preferred directions.

Analytical solutions are only known in the mean-field limit which corresponds to infinite dimensionality or, equivalently, infinite-range interactions. For the realistic case of short-ranged spin-glasses in finite dimensions d (= 3 in most physical applications) this may serve as a guideline, but for quantitative predictions we have to rely either on numerical methods such as Monte-Carlo simulations or on systematic expansion techniques such as high-temperature power series. The two approaches are quite complementary – each with its own drawbacks and merits.

Due to the competing interactions the phase space of spin-glasses is highly non-trivial with many important regions separated by high free-energy barriers. Monte-Carlo simulations are hence extremely difficult to equilibrate and the largest simulated systems are consequently usually quite small (of the order of 10^3 to 20^3). Refined update schemes such as multicanonical sampling [11], simulated and parallel tempering simulations [12,13] and the recently proposed multi-overlap method [14] target at this problem, but the numerical effort remains huge. Moreover, to model the quenched disorder properly, many replica (of the order of 100 to 10 000) with independently drawn random couplings have to be simulated. Since this is obviously an extremely time demanding task, scanning the two-dimensional parameter space of p and d is virtually impossible by this approach.

Using high-temperature series expansions, on the other hand, one can obtain for many quantities closed expressions in p and d up to a certain order in the inverse temperature. Here the infinite-volume limit can be taken without problems and the quenched disorder is treated exactly. Thus, by analyzing the resulting series, the behavior of spin-glasses can be monitored even as a continuous function of p and d. The main problem here is that the available series expansions are quite short (at any rate much shorter than for ferromagnetic systems). This introduces

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systematic errors of the resulting estimates which are difficult to control. The obvious way out is trying to extend the series as far as possible. This, however, is extremely cumbersome since the number of algebraic manipulations necessary to calculate the series coefficients blows up dramatically with the order of the series (usually at least exponentially).

Using an automatized star-graph expansion program package, we recently succeeded to extend the known series expansions [15–17] for the free energy and the "Edwards-Anderson" spin-glass susceptibility of the short-range p-state Potts glass model on general d-dimensional hypercubic lattices by one additional term to order K^{22} [18,19]. Here $K \equiv J\beta \equiv J/k_BT$ denotes the inverse temperature where k_B is the Boltzmann constant and $J > 0$ is the coupling strength of quenched ferro- and antiferromagnetic nearest-neighbor "exchange constants" $J_{ij} = \pm J$, which are randomly drawn from a symmetric bimodal distribution. In this paper we discuss the quite extensive analysis of the new series expansions in a large region of the two-dimensional (p, d)-parameter space. The flexibility of scanning a two-dimensional parameter region enables us to get an overview of the lower and upper critical dimensions of this model glass.

The rest of the paper is organized as follows. In Section 2, we briefly recall the model and some of the theoretical mean-field predictions based on the replica-breaking formalism. The Section 3 is devoted to a summary of the analysis techniques used. The results of our analysis are presented in Section 4, and in Section 5 we conclude with final remarks and an outlook to future work.

2 The model

The Hamiltonian of the p-state Potts glass model is defined as [5–9,20–22]

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij}\delta_{S_iS_j}, \tag{1}$$

where the spins S_i located at the sites i of a d-dimensional hypercubic lattice can take the p discrete values $S_i = 1, 2, \dots, p$, the symbol $\langle ij \rangle$ indicates nearest-neighbor interactions, $\delta_{S_iS_j}$ is the usual Kronecker delta symbol, and the "exchange constants" (bonds) J_{ij} are quenched, random variables. In the following we consider a bimodal distribution function

$$P_b(J_{ij}) = x\delta(J_{ij} - J) + (1 - x)\delta(J_{ij} + J), \tag{2}$$

where x denotes the concentration of ferromagnetic bonds and $J > 0$ their strengths. We furthermore specialize to the symmetric case $x = 1/2$. In references [18,19] high-temperature series expansions are derived for the free energy,

$$\beta F = -\left[\ln\left(\sum_{\{S_i\}} \exp(-\beta\mathcal{H})\right)\right]_{av}, \tag{3}$$

and the "Edwards-Anderson" (EA) susceptibility

$$\chi = \lim_{V\to\infty} \frac{1}{V} \sum_{i,j=1}^{V} \left[\left\langle \frac{1}{(p-1)} \left(p\delta_{S_iS_j} - 1\right)\right\rangle_T^2\right]_{av}, \tag{4}$$

where the angular brackets $\langle \dots \rangle_T$ refer to thermal averaging and the square brackets $[\dots]_{av}$ denote the average over the quenched, random disorder.

The high-temperature series expansion method gives the free energy and the susceptibility in the form

$$\beta F = 1 + a_8K^8 + a_{10}K^{10} + a_{12}K^{12} + \dots + a_{22}K^{22} + \dots, \tag{5}$$

and

$$\chi = 1 + b_2K^2 + b_4K^4 + b_6K^6 + \dots + b_{22}K^{22} + \dots, \tag{6}$$

with $K = J/k_BT$. The coefficients a_i and b_i depend on both p and d. Notice that due to the averaging over the symmetric quenched, random disorder no odd powers of K occur in the expansions (5, 6).

A useful consistency test of the series expansion for χ is the inversion method which yields a systematic large-d expansion of the transition point K_c [23]. Since in a second-order phase transition χ diverges at K_c, one may solve

$$1/\chi(K) = 0 \tag{7}$$

recursively with the ansatz

$$K^2 = K_c^2 = (p^2/2d)\left(1 + \sum_{i=1} c_i \left(\frac{1}{2d}\right)^i\right). \tag{8}$$

This yields the power-series expansions in $1/d$ for K_c^2 collected in Table 1 for several values of p.

Mean-field theory of the p-state Potts glass predicts for $p \leq 4$ a second-order freezing transition and for $p > 4$ a peculiar first-order phase transition with a jump of the "Edwards-Anderson" order parameter, but smooth moments and a vanishing latent heat [6–8]. For the infinite-range Potts glass (where $\sum_{\langle ij \rangle} \longrightarrow \sum_{i,j}$ in Eq. (1)) with $p = 3$ and $p = 6$ this scenario has recently been confirmed in quite extensive Monte-Carlo simulations [24–26]. The critical exponent of the susceptibility at the continuous transition is predicted by mean-field theory to be $\gamma_{MF} = 1$.

3 Series analysis techniques

In the literature many different series analysis techniques have been discussed which all have their own merits and drawbacks [27]. In general it is difficult to assess the accuracy of a given method when applied to relatively short series expansions. As a way out of this problem with systematic errors we repeated the analysis with several different analysis techniques which will be described next.

Table 1. Expansion coefficients c_i of the large-d expansion of the critical couplings K_c^2 of the p-state Potts glass model, $K_c^2 = (p^2/2d)\left[1 + c_1(1/2d) + c_2(1/2d)^2 + \ldots\right]$.

$c_i \setminus p$	2	3	4	5	6	8
c_1	$\frac{5}{3}$	$\frac{7}{4}$	$\frac{2}{3}$	$-\frac{19}{12}$	-5	$-\frac{46}{3}$
c_2	$\frac{443}{45}$	$\frac{1057}{80}$	$\frac{158}{45}$	$-\frac{3035}{144}$	$-\frac{293}{5}$	$-\frac{6562}{45}$
c_3	$\frac{394}{7}$	$\frac{102667}{1120}$	$\frac{12224}{105}$	$\frac{81887}{224}$	$\frac{9802}{7}$	$\frac{353216}{35}$
c_4	$\frac{676988}{1575}$	$\frac{4793401}{6400}$	$\frac{1005308}{1575}$	$-\frac{12026449}{16128}$	$-\frac{2770508}{175}$	$-\frac{768235012}{1575}$
c_5	$\frac{7620925}{2079}$	$\frac{376949671}{56320}$	$\frac{66400574}{10395}$	$-\frac{77969113625}{2128896}$	$-\frac{216036641}{385}$	$-\frac{229427448274}{10395}$

To simplify the notation we denote a thermodynamic function generically by $F(z)$ and assume that its Taylor expansion around the origin is known up to the Nth order,

$$F(z) = \sum_{n=0}^{N} a_n z^n + \ldots \tag{9}$$

If the singularity of $F(z)$ at the critical point z_c is of the simple form $(z \leq z_c)$

$$F(z) = A(z_c - z)^{-\lambda} + \ldots , \tag{10}$$

with A being a constant, then the logarithmic derivative of $F(z)$ exhibits a simple pole at $z = z_c$ with residue $-\lambda$,

$$\frac{d}{dz} \ln F(z) = \frac{\lambda}{z_c - z} + \ldots \tag{11}$$

This functional form is well-suited for an analysis by means of Padé approximants [28],

$$\begin{aligned} G(z) \approx [L/M] &\equiv \frac{P_L(z)}{Q_M(z)} \\ &\equiv \frac{p_0 + p_1 z + p_2 z^2 + \ldots + p_L z^L}{1 + q_1 z + q_2 z^2 + \ldots + q_M z^M}, \end{aligned} \tag{12}$$

where $L + M \leq N - 1$. Note than one order of the initial series is lost due to the differentiation in (11). It is well-known [28] that for a large class of functions, the so-called Stieltjes functions, the residues of the diagonal and next-to-diagonal Padé sequences $[N/N]$ and $[N/N \pm 1]$ converge to the true critical exponent λ. This is the widely used Dlog-Padé method.

The dots in (10, 11) indicate correction terms which can be parameterized as follows:

$$F(z) = A(z_c - z)^{-\lambda}\left[1 + A_1(z_c - z)^{\Delta_1} + A_2(z_c - z) + \ldots\right], \tag{13}$$

where Δ_1 is the confluent correction exponent and the second term is a usually weaker analytic correction. Such a more general critical behavior can be analyzed with two related methods discussed in detail in reference [29].

In the method referred to as M1, first the leading singularity is removed by forming

$$\begin{aligned} B &\equiv \lambda F(z) - (z_c - z)\frac{d}{dz}F(z) \\ &= A(z_c - z)^{-\lambda} \\ &\quad \times \left[\Delta_1 A_1 (z_c - z)^{\Delta_1} + A_2(z_c - z) + \ldots\right]. \end{aligned} \tag{14}$$

Then Padé approximants are applied to the logarithmic derivative of B,

$$\frac{d}{dz}\ln B = \frac{A_1\Delta_1(\lambda - \Delta_1)(z_c - z)^{\Delta_1 - 1} + A_2(\lambda - 1)}{(z_c - z)(A_1\Delta_1(z_c - z)^{\Delta_1 - 1} + A_2)}, \tag{15}$$

yielding for fixed z_c the confluent correction exponent Δ_1 as a function of λ, $\Delta_1 = \Delta_1(\lambda)$. The optimal set of values for the parameters z_c, γ, and Δ_1 is determined visually from the best clustering of different Padé approximants.

In the second method referred to as M2, Padé approximants in a new variable (Roskies transformation),

$$y = 1 - (z_c - z)^{\Delta_1}, \tag{16}$$

are applied to

$$\begin{aligned} G(y) &\equiv -\Delta_1(1 - y)\frac{d}{dy}\ln F \\ &= -\lambda + \frac{A_1\Delta_1(z_c - z)^{\Delta_1} + A_2(z_c - z)}{1 + A_1(z_c - z)^{\Delta_1} + A_2(z_c - z)} \\ &= -\lambda + \frac{A_1\Delta_1(1 - y) + A_2(1 - y)^{1/\Delta_1}}{1 + A_1(1 - y) + (1 - y)^{1/\Delta_1}}, \end{aligned} \tag{17}$$

yielding for fixed z_c the critical exponent λ as a function of Δ_1, $\lambda = \lambda(\Delta_1)$. Again the clustering of different Padé approximants is used to select the optimal set of parameters. The two methods are complementary and as stressed in Appendix D of reference [29] should always be used in conjunction to avoid spurious results due to so-called resonances at values of Δ_1/n, $n = 2, 3, \ldots$ in the otherwise more accurate method M2.

Another generalization of Padé approximants are differential Padé approximants (DPA) [27]. Here one starts from the usual Dlog-Padé method,

$$\frac{d}{dz}\ln F(z) = \frac{F'(z)}{F(z)} = \frac{P_L(z)}{Q_M(z)} + \mathcal{O}(z^{L+M+2}), \qquad (18)$$

and rewrites this in the form of a differential equation,

$$Q_M(z)F'(z) - P_L(z)F(z) = 0. \qquad (19)$$

This suggested a generalization to [27]

$$\sum_{i=0}^{K} Q_i(z)\left(z\frac{d}{dz}\right)^i F(z) = S_L(z), \qquad (20)$$

where $Q_i(z) = \sum_{j=0}^{M_i} Q_{i,j}z^j$, $S_L(z) = \sum_{j=0}^{L} S_j z^j$, and $\sum_{i=0}^{K} M_i + K + L \le N$. The inhomogeneity S_L can account for additive analytic terms in $F(z)$. In the present analysis we employed three special cases:

$$Q_1(z)zF'(z) + Q_0F(z) = S(z) \qquad \text{(DPA1)}, \quad (21)$$

$$Q_2(z)z^2F''(z) + Q_1(z)zF'(z) + Q_0F(z) = 0 \qquad \text{(DPA2)}, \quad (22)$$

$$Q_2(z)z^2F''(z) + Q_1(z)zF'(z) + Q_0F(z) = S(z) \qquad \text{(DPA3)}. \quad (23)$$

The more elaborate DPA analysis was mainly used to double-check the methods M1 and M2 whose implementation is much more convenient [30].

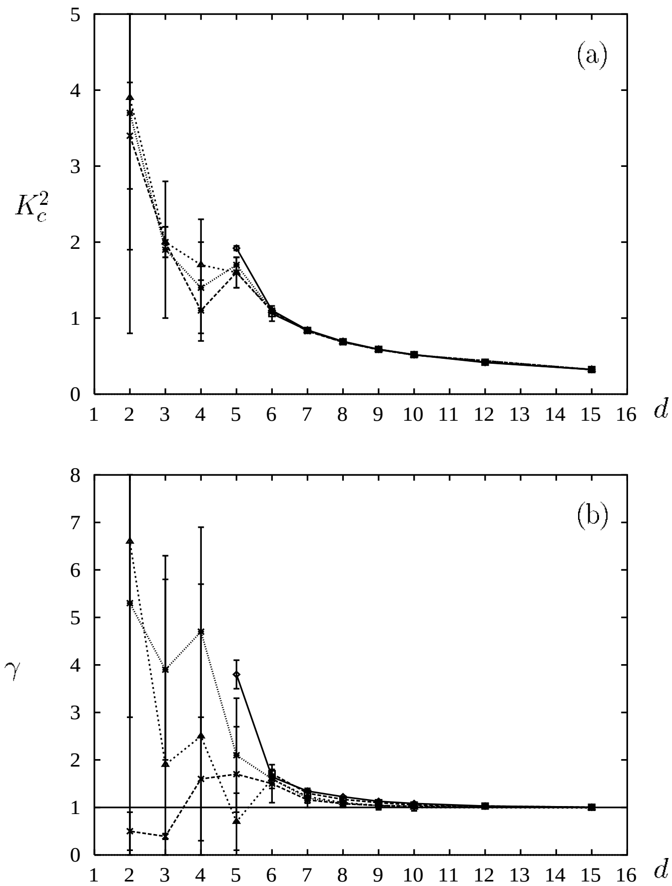

Fig. 1. Comparison of the analysis methods for $p = 3$: (◇): M1/M2; (□): Dlog-Padé; (×): DPA1; (△): DPA2; (∗): DPA3.

4 Results

The series expansions of references [18,19] are given for arbitrary values of p and d. In principle we could, therefore, scan the whole continuous two-dimensional parameter space (p, d) with the series analysis. Even though our analysis methods are highly automatized and supported by graphical tools a quasi-continuous scanning would be, however, still quite an elaborate task. Since it is also questionable if the additional information for non-integer values of p and d would be really helpful, we confined ourselves to a grid of integer tuples (p, q) in the range $p = 2, \ldots, 8$ and $d = 2, \ldots, 15$.

For each parameter tuple we applied the different analysis techniques described in Section 3. In this way we reduced the danger of picking up systematic errors which may arise due to the finiteness of the series expansions (and which is sometimes difficult to detect intrinsically). As our final results we thus usually quote a weighted average over the different methods, and the error bars are estimated from the variation among the different estimates.

As an illustration of this procedure we compare in Figure 1 the different methods for $p = 3$ and $d = 2, \ldots, 15$. Even with the extended series expansion up to the order K^{22} we observe a clear tendency that for d smaller than 4 or 5 the estimates for K_c and γ start to become quite unreliable. While for K_c the scatter of the data is still moderate, the critical exponent estimates do depend strongly on the method used. This is the reason why in most of the following plots we will exclude the points for $d \le 4$.

By performing the same type of analysis for $p = 2, 4, 5, 6$, and 8 we obtained the data for K_c and γ collected in Tables 2 and 3 and graphically presented in Figures 2 and 3. Let us first discuss the transition points K_c. For large d we can compare them with the large-d expansion discussed in Section 2. The resulting curves (using all terms in Tab. 1) are shown in Figure 4. While for $p = 2$, 3, and 4 the agreement is satisfactory down to about $d = 5$, for $p = 6$ the $1/d$-expansion curves down already for relatively large d due to the negative sign of the expansion coefficients. As will become clearer in the discussion of the critical exponent γ, this may be taken as an indication that for $p > 4$ the freezing transition is, in fact, of first-order. For $p > 4$ we are thus determining *effective* exponents and transition temperatures, which should be related to the boundary of the metastability region at a first-order phase transition. In general the $1/d$-expansion of K_c is expected to be an asymptotic series which, for any given finite number of terms should approach the exact answer as $1/d \longrightarrow 0$. For small d, on the other hand, it is not expected to converge as the length of the expansion is increased. In fact, it is

Table 2. Critical couplings K_c^2 of the p-state Potts glass in d dimensions.

$d \backslash p$	2	3	4	5	6	8
5	0.606(3)	1.92(3)	2.20(3)	2.30(2)	2.50(5)	2.560(2)
6	0.4399(1)	1.101(1)	1.56(1)	1.870(1)	2.05(1)	2.235(5)
7	0.3520(5)	0.843(1)	1.310(5)	1.630(5)	1.790(5)	1.995(5)
8	0.2945(5)	0.694(1)	1.111(2)	1.410(5)	1.5859(1)	1.769(3)
9	0.2550(1)	0.5897(2)	0.9744(1)	1.260(5)	1.441(1)	1.630(1)
10	0.2246(1)	0.5165(5)	0.8593(1)	1.138(2)	1.323(1)	1.529(3)
12	0.1822(1)	0.416(1)	0.7007(2)	0.960(1)	1.149(1)	1.360(4)
15	0.1426(1)	0.3235(5)	0.5502(2)	0.774(1)	0.9610(3)	1.19(2)

Table 3. Critical exponents γ of the susceptibility of the p-state Potts glass in d dimensions.

$d \backslash p$	2	3	4	5	6	8
5	1.71(4)	3.8(3)	1.52(4)	0.985(5)	0.82(5)	0.463(4)
6	1.330(2)	1.643(5)	1.188(4)	0.930(1)	0.75(3)	0.461(4)
7	1.19(2)	1.345(5)	1.175(5)	0.97(4)	0.74(2)	0.454(6)
8	1.100(4)	1.226(2)	1.355(10)	0.93(1)	0.7147(3)	0.414(10)
9	1.07(1)	1.1294(4)	1.146(2)	0.93(3)	0.723(3)	0.408(4)
10	1.042(8)	1.0825(5)	1.0900(4)	0.935(10)	0.728(5)	0.426(4)
12	1.004(8)	1.03(2)	1.0475(2)	0.945(20)	0.767(2)	0.440(5)
15	1.00(1)	1.005(20)	0.99(1)	0.9310(5)	0.799(4)	0.49(2)

Table 4. Results for $p = 3$ of the $1/d$-expansion for K_c^2.

d	K_c^2	# terms
2	no convergence	
3	1.9	1
4	1.8	3
5	1.4	5
6	1.0	5
7	0.81	5
8	0.68	5
9	0.58	5
10	0.51	5
12	0.415	5
15	0.323	5

a well-known property of asymptotic series that for relatively large expansion parameter (small d in our case) a greater accuracy can be achieved if only a small number of terms is kept. Optimal estimates can usually be obtained by truncating the expansion after the smallest term. Our numerical results for $p = 3$ following this recipe are shown in Table 4.

Physically more interesting is the limit of small dimensions where the behavior of K_c determines the lower critical dimension d_l, *i.e.*, the dimension below which K_c tends to infinity. The transition from the disordered phase to the spin-glass phase then occurs only at zero temperature. In Figure 5 we have replotted the data of Figure 2 in the form $1/K_c^2$ *versus* d and included least-squares fits to the ansatz $1/K_c^2 = a_0 + a_1 d + a_2 d^2 + a_3 d^3$. From the crossing points with the dotted line at $1/K_c^2 = 0$ an estimate of d_l can be read off. The results are $d_l \approx 2.28$ for $p = 2$, 3.20 ($p = 3$), 1.81 ($p = 4$), -0.007 ($p = 5$), 1.06 ($p = 6$), and -1.81 ($p = 8$). While one may expect that the case $p = 2$ is special due to the spin-reversal symmetry of the Ising model, one would not expect that every p yields a different result for d_l, and the values for d_l found for $p \geq 5$ clearly are unphysical. Nevertheless for $p = 2$ and 3 our values are consistent with estimates from Monte-Carlo simulations. For $p = 3$ this corroborates the claim of reference [31] that the three-dimensional model is at its lower critical dimension where it should exhibit an essential (exponential type) singularity at zero temperature [32]. In contrast to reference [31], however, we feel that even the extended series expansions are still to short to warrant a more detailed investigation of this question. Our comparative study of d_l for many values of p shows that series analyses of d_l for such spin-glass models are doubtful, with the available number of terms, contrary to claims in the literature!

Let us now turn to a discussion of the critical exponent γ. In Figure 3a we observe for any dimension $d \geq 5$ a clear qualitative distinction between the cases $p \leq 4$ and $p > 4$. While we obtain $\gamma \geq 1$ for $p = 2$, 3, and 4, we find $\gamma < 1$ for $p = 5$, 6, and 8. In order to understand this qualitative difference we performed a

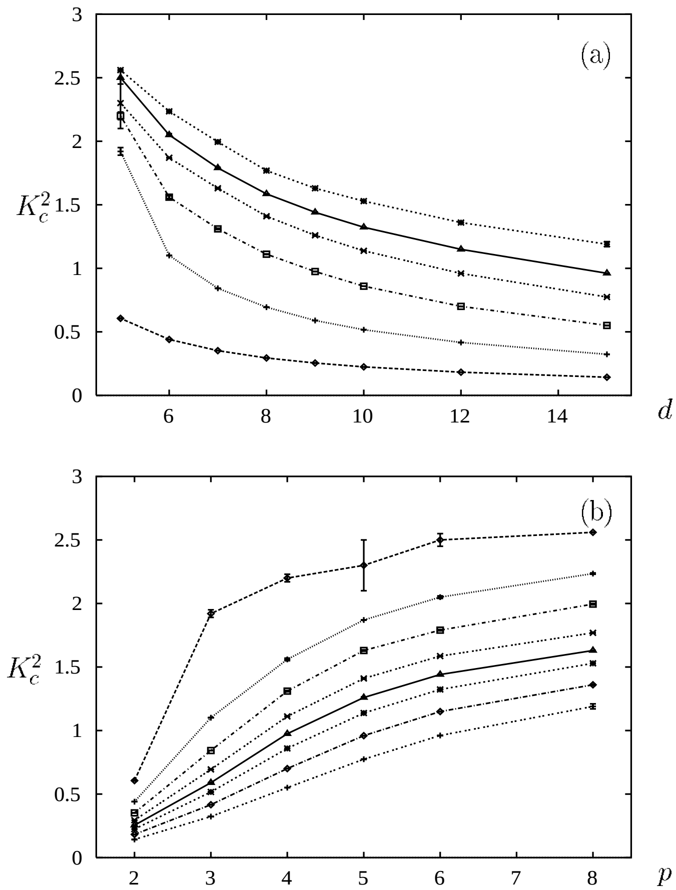

Fig. 2. The critical coupling K_c^2 (a) as a function of the dimension d: (◇): $p = 2$; (+): $p = 3$; (□): $p = 4$; (×): $p = 5$; (△): $p = 6$; (∗): $p = 8$; (b) as a function of the number of Potts states p; from top to bottom: (◇): $d = 5$; (+): $d = 6$; (□): $d = 7$; (×): $d = 8$; (△): $d = 9$; (∗): $d = 10$; (◇): $d = 12$; (+): $d = 15$.

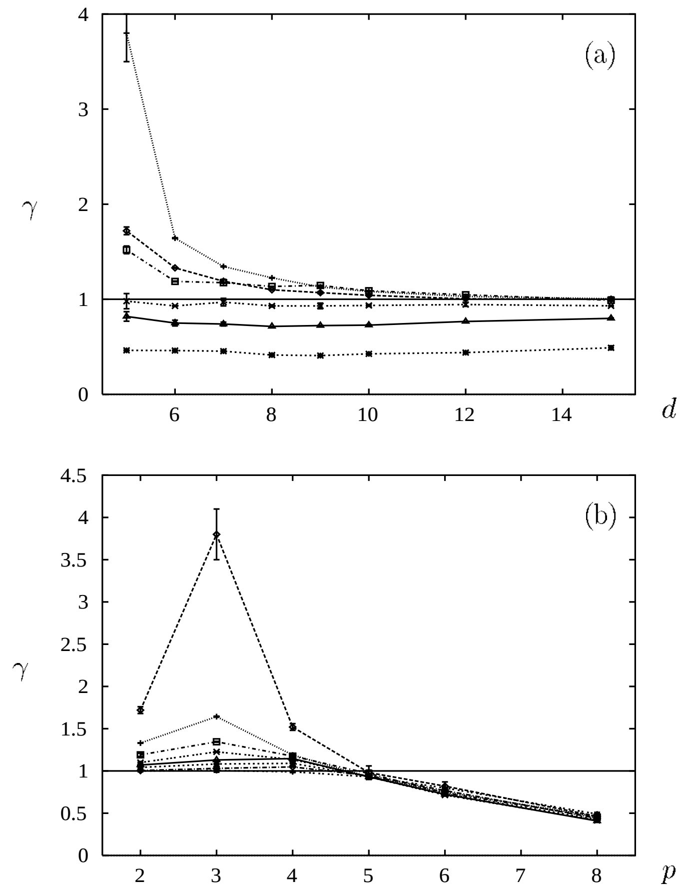

Fig. 3. The critical exponent γ of the susceptibility χ (a) as a function of the dimension d: (◇): $p = 2$; (+): $p = 3$; (□): $p = 4$; (×): $p = 5$; (△): $p = 6$; (∗): $p = 8$; (b) as a function of the number of Potts states p; from top to bottom: (◇): $d = 5$; (+): $d = 6$; (□): $d = 7$; (×): $d = 8$; (△): $d = 9$; (∗): $d = 10$; (◇): $d = 12$; (+): $d = 15$.

comparative series analysis of the pure ferromagnetic p-state Potts model, where the nature of the phase transition is known quite reliably from other numerical sources (and in $d = 2$ exactly). From this study it became clear that $\gamma < 1$ should be interpreted as an *effective* exponent, signalizing the metastability boundary at a *first-order* phase transition (recall the discussion of K_c). We thus interpret our data as evidence for a first-order phase transition in the short-range p-state Potts glass for $p > 4$ and $d \geq 5$. At this point it is interesting to recall that mean-field theory of Potts glasses does indeed predict [6–8] a new, unusual kind of first-order phase transition for $p > 4$, without latent heat and a part of the order-parameter distribution function that appears discontinuously at T_c. Although this type of transition significantly differs from standard first-order transitions as they occur in the Potts ferromagnet, one does expect that one should be able to detect these transitions from a high-temperature series analysis as well, since the nature of this transition is much closer to a second-order transition than that of the corresponding Potts ferromagnet. Unfortunately a more thorough series analysis of the conjectured first-order phase transitions is highly nontrivial since the necessary technical tools are

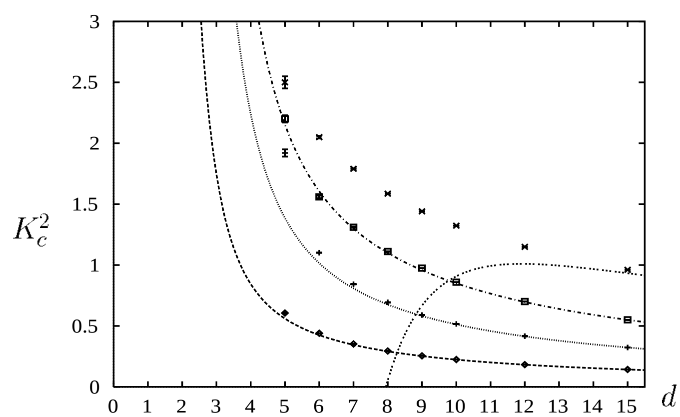

Fig. 4. Comparison of the large-d expansions for K_c^2 (dashed lines) with directly obtained estimates: (◇): $p = 2$; (+): $p = 3$; (□): $p = 4$; (×): $p = 6$.

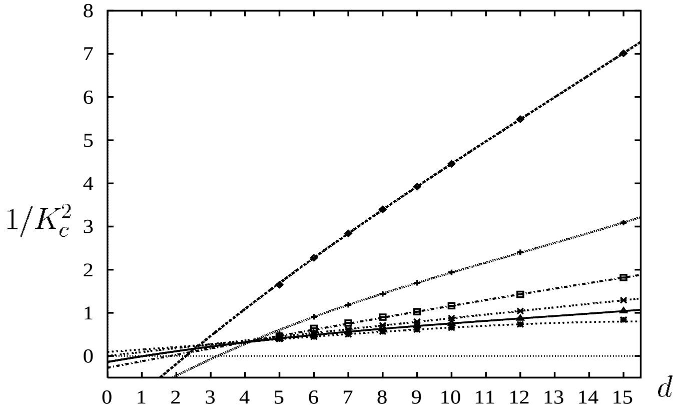

Fig. 5. Least-squares fits of the inverse critical couplings to the ansatz $1/K_c^2 = a_0 + a_1 d + a_2 d^2 + a_3 d^3$: (◇): $p = 2$; (+): $p = 3$; (□): $p = 4$; (×): $p = 5$; (△): $p = 6$; (∗): $p = 8$.

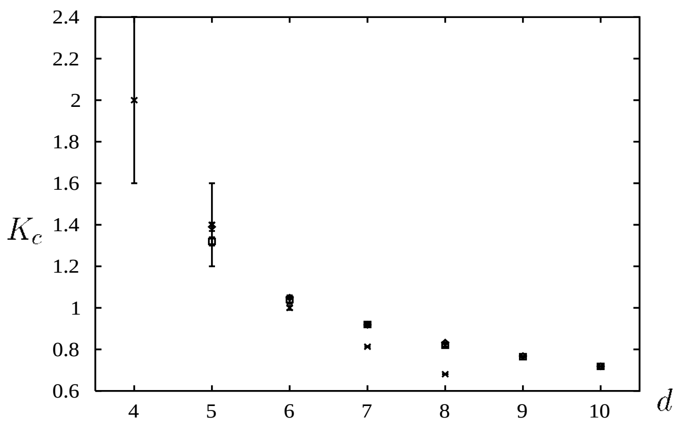

Fig. 6. Critical couplings K_c for $p = 3$. Comparison of the present results (◇) with the high-temperature series analysis of reference [31] (□) and reference [33] (×).

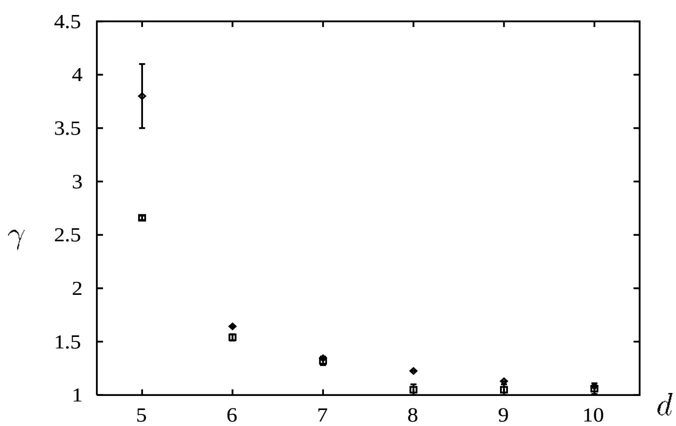

Fig. 7. Critical exponents γ for $p = 3$. Comparison of the present results (◇) with previous estimates of reference [31] (□).

not yet well-established for this type of transition (the response functions exhibit jumps at the transition rather than power-law singularities, as in the case of a second-order transition). Accepting thus the $\gamma < 1$ criterion, we see in Figure 3b a clear crossover between the second- and first-order phase transition regimes between $p = 4$ and 5 for all investigated dimensions d.

Focusing now on the cases $p = 2$, 3, and 4, the large-d behavior of γ determines the upper critical dimension d_u at which γ attains the mean-field value of $\gamma_{MF} = 1$. By looking at Figure 3a it is obvious that, with the available series expansions, the only safe statement one can make is that $d_u \leq 15$. This is certainly not very predictive, but we do not see any decisive difference between the γ-values at $d = 6$ (the commonly accepted value of d_u), or $d = 8$ (the value of d_u tentatively supported in Ref. [31]), or even $d = 10$. We rather see in this range of dimensions a smooth crossover with all estimates for γ being clearly greater than unity. This is very likely an artifact of the series expansion method, caused by the fact that the available series expansions are still rather short. Only with considerably extended series expansion one may have a chance to determine d_u more accurately. It should be noted that these strong deviations from mean-field behavior at very high dimensionalities in these series analyses are rather unusual, for Ising ferromagnets one has little difficulties to verify mean-field behavior at high dimensionalities with very short series, for instance.

Since this conclusion is in apparent disagreement with the claims of previous studies for $p = 2$ and 3 we have examined these special cases in greater detail. Our results for $p = 3$ are compared in Figures 6 and 7 with previous series estimates of references [31,33]. We first notice that the estimates for K_c of reference [33] clearly deviate. This, however, is expected since already the underlying series expansions of reference [33] do not agree with ours. We emphasize that our series expansion of χ successfully passed the inversion test which is a necessary and usually quite stringent (albeit not sufficient) condition that the expansion is correct. The agreement with the estimates for K_c of reference [31] (for more details, see Ref. [15]), on the other hand, is almost perfect. Also this is not too surprising since the analysis in reference [31] is based on series expansions with only one term less than the present ones (and all the others agree). Still, when comparing the more sensitive critical exponent γ we find some differences which are particularly pronounced at $d = 5$. As far as the upper critical dimension d_u is concerned, the differences at $d = 8$ and 9 are more important, however. Here the estimates of reference [31] are significantly lower than ours. It is then, in fact, tempting to speculate that γ has approached unity at $d = 8$, but differs from unity for $d < 8$, as was concluded in reference [31]. With the present data, on the other hand, it is clear that such a claim that $d = 8$ is a special dimension is not justified at all.

For $p = 2$ the Potts glass degenerates to the much simpler and hence more extensively studied "Edwards-Anderson" Ising spin-glass. In Figures 8 and 9 we compare our results for K_c and γ with previous estimates in

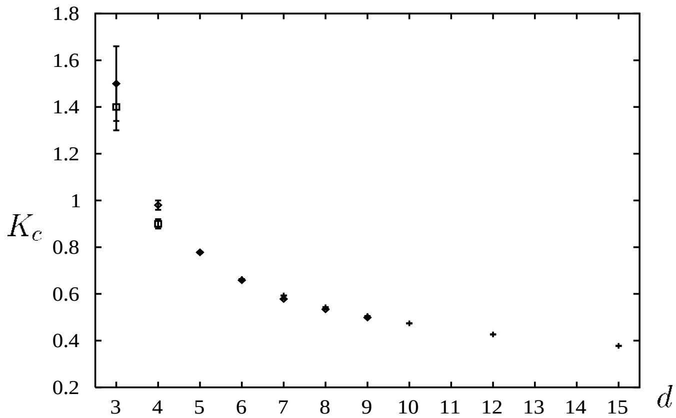

Fig. 8. Critical couplings K_c for $p = 2$. Comparison of the present results (+) with the high-temperature series analysis of reference [29] (◇) and references [34–36] (□).

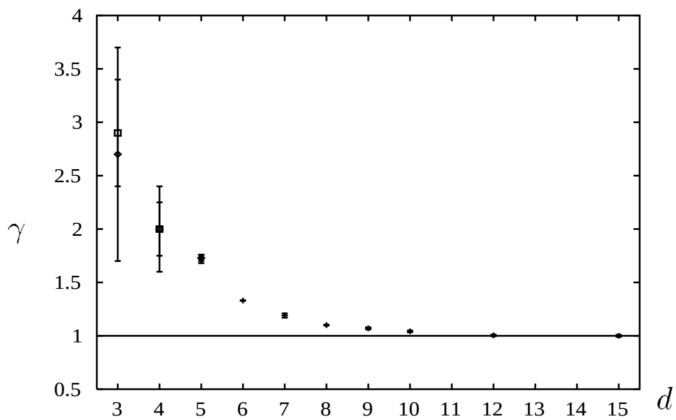

Fig. 9. Critical exponents γ for $p = 2$. Comparison of the present results (+) with previous results of reference [29] (◇) and references [34–36] (□).

references [29] and [34–37] (using longer series derived for the special case of the Ising spin-glass model[1]). Here we find very good agreement with our results for both the critical coupling K_c and the critical exponent γ. This indicates that the analysis methods are well under control. But also here we would hesitate to make any strong statement about the upper critical dimension d_u (other than $d_u \leq 12$).

5 Discussion

We have analyzed recently extended high-temperature series expansions for the susceptibility of general p-state Potts glass models defined on hypercubic lattices in arbitrary dimensions d. By scanning the two-dimensional (p, d) parameter space we obtain a qualitative overview over the properties of this glass model. In particular, the critical coupling is obtained quite reliably for a wide range of these parameters, and this should be a useful check for other methods, *e.g.*, if one attempts to do Monte-Carlo simulations for high-dimensional lattices. For $p > 4$ and $d \geq 5$ our analysis suggests a first-order freezing transition in agreement with predictions of mean-field theory. An accurate estimation of lower and upper critical dimensions turned out to be hardly possibly with the relatively short series expansions (up to K^{22}) at hand. In particular we cannot confirm the claim of reference [31] that $d_u = 8$ for the model with $p = 3$. We feel that the sharp change of γ at $d = 8$ is an artifact of a somewhat incomplete analysis, and cannot be maintained in the light of the present results. Our analysis rather shows that, due to the finiteness of the series expansions, a rather smooth crossover from $\gamma > 1$ to $\gamma = 1$ occurs in the range $d \approx 6-12$. The available series expansions are still too short to read off a definite trend with increasing order. Here longer series may be of considerable help, and we are currently pursuing this line of approach with a modified expansion scheme.

Even though quantitative predictions are still somewhat limited, we are convinced that the high-temperature series expansion approach to complex physical systems will continue to be a worthwhile and complementary alternative to other methods such as, *e.g.*, numerical simulations. The overhead of generating the series expansions to sufficiently high order may appear huge, but eventually this investment may pay off. Among the main advantages is the possibility to scan a whole two- or higher-dimensional parameter space without too much labor once the expansions are known. Other points worth mentioning in the context of spin-glasses are: quenched averages can be performed exactly, the thermodynamic limit is always implied, and the whole phase space is properly taken into account such that no non-ergodicity problems can affect equilibrium quantities.

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[1] For the comparison with the $p = 2$ Potts glass it should be noted that the Ising Hamiltonian is usually taken as $\mathcal{H}_{IS} = -\sum_{\langle ij \rangle} J_{ij} s_i s_j$ with $s_i = \pm 1$, leading to the conversion $K_{IS} = K/2$, and that in the Ising formulation the expansion variable is often chosen as $w \equiv (\tanh K_{IS})^2$.

References

1. K. Binder, A.P. Young, Rev. Mod. Phys. **58**, 801 (1986).
2. M. Mezard, G. Parisi, M. Virasoro, *Spin-Glass Theory and Beyond* (World Scientific, Singapore, 1987).
3. K.H. Fischer, J. Hertz, *Spin-Glasses* (Cambridge University Press, Cambridge, 1991).
4. D.S. Stein, *Spin-Glasses and Biology* (World Scientific, Singapore, 1992).
5. D. Elderfield, D. Sherrington, J. Phys. C **16**, L491 (1983); *ibid.* L971 (1983); *ibid.* L1169 (1983).
6. D.J. Gross, I. Kanter, H. Sompolinsky, Phys. Rev. Lett. **55**, 304 (1985).

7. G. Cwilich, T.R. Kirkpatrick, J. Phys. A **22**, 4971 (1989).
8. G. Cwilich, J. Phys. A **23**, 5029 (1990).
9. K. Binder, J.D. Reger, Adv. Phys. **41**, 547 (1992).
10. U.T. Höchli, K. Knorr, A. Loidl, Adv. Phys. **39**, 405 (1990).
11. B.A. Berg, T. Celik, Phys. Rev. Lett. **69**, 2292 (1992); B.A. Berg, U. Hansmann, T. Celik, Phys. Rev. B **50**, 16444 (1994).
12. E. Marinari, G. Parisi, J.J. Ruiz-Lorenzo, in *Spin-Glasses and Random Fields*, edited by A.P. Young (World Scientific, Singapore, 1998), p. 59.
13. K. Hukusima, K. Nemoto, J. Phys. Soc. Jpn **65**, 1604 (1996).
14. B.A. Berg, W. Janke, Phys. Rev. Lett. **80**, 4771 (1998).
15. G. Schreider, Ph.D. thesis, Universität Mainz (1993).
16. G. Schreider, J.D. Reger, J. Phys. A **27**, 1071 (1994).
17. G. Schreider, J.D. Reger, J. Phys. A **27**, 1079 (1994).
18. B. Lobe, Ph.D. thesis, Universität Mainz (1997).
19. B. Lobe, W. Janke, Mainz preprint (in preparation).
20. R. Pirc, B. Tadic, Phys. Rev. B **54**, 7121 (1996).
21. K. Binder, in *Spin-Glasses and Random Fields*, edited by A.P. Young (World Scientific, Singapore, 1998), p. 99.
22. B.O. Peters, B. Dünweg, K. Binder, M. d'Onorio de Meo, K. Vollmayr, J. Phys. A **29**, 3503 (1996).
23. M.E. Fisher, R.R.P. Singh, in *Disorder in Physical Systems*, edited by G.R. Grimmett, D.J.A. Welsh (Oxford University Press, Oxford, 1990), p. 87.
24. O. Dillmann, Diploma thesis, Universität Mainz (1997).
25. O. Dillmann, W. Janke, K. Binder, J. Stat. Phys. **92**, 57 (1998).
26. O. Dillmann, W. Janke, K. Binder, Mainz preprint (February 1998), to appear in *Computer Simulation Studies in Condensed-Matter Physics XI*, edited by D.P. Landau, H.-B. Schüttler (Springer, Berlin, 1999).
27. A.J. Guttmann, in *Phase Transitions and Critical Phenomena*, Vol. 13, edited by C. Domb, J.L. Lebowitz (Academic Press, New York, 1989), p. 1.
28. G.A. Baker Jr., *Essentials of Padé Approximants* (Academic Press, London, 1975).
29. L. Klein, J. Adler, A. Aharony, A.B. Harris, Y. Meir, Phys. Rev. B **43**, 11249 (1991).
30. J. Adler, I. Chang, S. Shapira, Int. J. Mod. Phys. C **4**, 1007 (1993); J. Adler, Comp. Phys. **8**, 1 (1994).
31. G. Schreider, J.D. Reger, J. Phys. A **28**, 317 (1995).
32. W.L. McMillan, J. Phys. C **17**, 3179 (1984).
33. R.R.P. Singh, Phys. Rev. B **43**, 6299 (1991).
34. R.R.P. Singh, S. Chakravarty, Phys. Rev. Lett. **57**, 245 (1986).
35. R.R.P. Singh, S. Chakravarty, Phys. Rev. B **36**, 546, 559 (1987).
36. R.R.P. Singh, S. Chakravarty, J. Appl. Phys. **61**, 4095 (1987).
37. R.R.P. Singh, M.E. Fisher, J. Appl. Phys. **63**, 3994 (1988).